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Problem. (**Proposed by Daniel Sitaru-Romania**).

In acute $\triangle ABC$ the following relationship holds:

$$\frac{a\cos A}{b\cos B}+\frac{b\cos B}{c\cos C}+\frac{c\cos C}{a\cos A}\le\frac{3}{8\cos A\cos B\cos C}.$$

Solution by Arkady Alt, **San Jose**,**California**, **USA**.

Since $a:b:c=\sin A:\sin B:\sin C$ then $\sum\frac{a\cos A}{b\cos B}=\sum\frac{2R\sin A\cos A}{2R\sin B\cos B}=\sum\frac{\sin 2A}{\sin 2B}$.

Denoting $\alpha:=\pi-2A,\beta:=\pi-2B,\gamma:=\pi-2C$ we obtain $\alpha+\beta+\gamma=\pi$ and $\alpha,\beta,\gamma>0$ (because $A,B,C<\pi/2$). Then, $\sum\frac{a\cos A}{b\cos B}\le\frac{3}{8\cos A\cos B\cos C}\Leftrightarrow$

(**1**) $\sum\frac{\sin\alpha}{\sin\beta}\le\frac{3}{8\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}$.

For further let $\triangle ABC$ be any fixed triangle (not to be confused with acute triangle ABC from the problem statement) with angles α,β,γ opposite sides BC,CA,AB, respectively and let a,b,c,s,R and r be standard notation for sidelengths, semiperimeter, circumradius and inradius, respectively.

Since $r=4R\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$ and $\sum\frac{\sin\alpha}{\sin\beta}=\sum\frac{2R\sin\alpha}{2R\sin\beta}=\sum\frac{a}{b}$ then (**1**)$\Leftrightarrow$

(**2**) $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\le\frac{3R}{2r}$.

Thus, the proof of inequality of the problem for any acute triangles equivalently reduced to the proof of inequality (**2**) for any triangle.

Since by Cauchy Inequality $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\le\sqrt{a^2+b^2+c^2}\cdot\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}$

then for completing solution of the problem remains to prove inequality

(**3**) $(a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\le\frac{9R^2}{4r^2}$.

We have $\sum\frac{1}{a^2}=\sum\frac{h_a^2}{4F^2}=\frac{1}{4r^2s^2}\sum h_a^2$, where $F=rs$ is area of $\triangle ABC$ and h_a,h_b,h_c be altitudes in $\triangle ABC$. Let l_a,l_b,l_c be lengths of angle bisectors from vertices A,B,C, respectively. Since $h_x^2\le l_x^2$ and $l_x^2\le s(s-x),\ x\in\{a,b,c\}$ then $\sum h_a^2\le\sum s(s-a)=s^2$.

Hence, $\sum\frac{1}{a^2}\le\frac{1}{4r^2s^2}\cdot s^2=\frac{1}{4r^2}$ and using well known inequality* $a^2+b^2+c^2\le 9R^2$ we finally obtain $(a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\le\frac{9R^2}{4r^2}$.

* Since $a^2+b^2+c^2=2(s^2-4Rr-r^2)$, $s^2\le 4R^2+4Rr+3r^2$(Gerretsen's Inequality) and $R\ge 2r$ (Euler's Inequality) we have

$9R^2-(a^2+b^2+c^2)=9R^2-2(s^2-4Rr-r^2)\ge 9R^2-2((4R^2+4Rr+3r^2)-4Rr-r^2)=$
$(R-2r)(R+2r)\ge 0$.

Another, short proof of inequality $a^2+b^2+c^2\le 9R^2$ based on using distance formula in baricentric geometry.